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1978 J. Phys. A: Math. Gen. 11 1583

(http://iopscience.iop.org/0305-4470/11/8/019)

J. Phys. A: Math. Gen. Vol. 11, No. 8, 1978. Printed in Great Britain

Thermal particle production in a radiation dominated Robertson–Walker universe

J Audretsch and G Schäfer
Fachbereich Physik der Universität Konstanz, Postfach 7733, D-7750 Konstanz, Federal Republic of Germany

Received 15 February 1978, in final form 12 April 1978

Abstract. Creaction of massive spin-$\frac{1}{2}$ particles in a 3-flat Robertson–Walker universe with expansion law $R \sim t^{1/2}$ (radiation dominated universe) is studied with Fock space methods. The universe is thereby completed in 'passing through the singularity' by a time-symmetric (mirror-like) contracting universe preceding the singularity. The respective procedure to do so for Dirac test fields (conformal method) is discussed in detail. For the asymptotic in- and out-regions a WKB particle interpretation is applied. It is found that particles are created with a non-relativistic thermal spectrum. The expressions for the number density, energy density and pressure of the created particles confirm this result (equation of state). The study of the development in time of the creation process shows that 99% of the particles created all together are created at about the Compton time.

1. Introduction

The main result of quantum field theory in curved space–time is that it leads to the creation of particles by strong gravitational fields. A variety of problems related herewith has been discussed during recent years mainly with respect to the gravitational fields represented by cosmological and black hole space–times. For a survey of the literature till 1975 we refer to De Witt (1975) and Parker (1977).

In this paper we are concerned with the creation of massive spin-$\frac{1}{2}$ particles caused by the expansion of an open spatially flat Robertson–Walker universe with expansion law $R \sim t^{1/2}$, which is that of a radiation dominated cosmology. In particular we are interested in the probability distribution, the average number density, the energy density, the pressure and the entropy of the created particles. Throughout the calculation the cosmological model will thereby be taken as a given underlying classical space–time which is otherwise determined and which is not modified by the created particles. Accordingly quantum fluctuations of the metric are excluded and the back-reaction of the created matter on the expansion is neglected. Furthermore, we will trace the history of our universe back to the states of infinite density and beyond that to the earlier states with finite density in the contracting 'mirror-like' universe before.

These simplifying assumptions have to be kept in mind when the results of this paper and those of similar calculations are used to discuss the 'true' cosmological situation during the first second of the universe. We have the disadvantage of the discussion of a highly idealised situation which on the other hand implies the advantage of rigorous results. Cosmology as part of astrophysics is the discussion of a highly

0305-4770/78/0008-1583$03.00

complex situation. But this discussion is based on results of 'pure cases'. So that in this sense the results of quantum field theory in given cosmological space–times can be of fundamental cosmological importance.

Particle creation in Robertson–Walker universes has been studied by several authors. The respective treatments mainly differ in the choice of the particle wave equation, in the definition of the vacuum and in the way the Einstein expansion law $R(t)$ with singularity is changed. Parker (1971) modifies $R(t) \sim t^{1/2}$ by assuming that $R(t)$ approaches a constant value for $t \leqslant 0$ and $t \geqslant t_2$. He then obtains for Dirac particles that the density of all created particles is finite. Mamaev *et al* (1976) study bosons in a Friedman universe with $R \sim t^q$, $0<q<1$ and define the vacuum state at any time t as well as $t=0$, by instantaneous diagonalisation of the energy–momentum tensor and time-dependent normal ordering†. The most interesting result with regard to our calculations is that the virtual particles in the vacuum are obtained with a thermal spectrum, while this is not the case for the created real particles. Parker (1976) studies massless minimally coupled spin-0 particles and the expansion law $R \sim t^{1/2}$ which is modified at about the Planck time in joining it to a constant value of R. He obtains approximately a black-body distribution for the created particles. It is not yet clear how this result depends on the modification of the $R \sim t^{1/2}$ law at early times. Chitre and Hartle (1977) define for conformally coupled massive bosons and expansion law $R \sim t$ an initial vacuum state at the singularity in applying a path integral formulation. This is equivalent to imposing certain boundary conditions at the singularity. But there is no physical interpretation of these conditions. Furthermore for a behaviour $R \sim t^q$, $0<q \leqslant 1$ at $t=0$, only $q=1$ implies that the conformal transformation of the time-coordinate $t \to \eta$, relating to Minkowski space–time, transforms the singularity at $t=0$ to $\eta=-\infty$. The spectrum of produced particles in Chitre and Hartle (1977) becomes a thermal spectrum for high energies.

In contrast to the approaches above we pass through the big bang singularity into a 'mirror-like' contracting universe and obtain for spin-$\frac{1}{2}$ particles and an unmodified expansion law $R \sim t^{1/2}$ a thermal spectrum for all energies. For the asymptotic in- and out-region a WKB particle interpretation is applied.

In § 2 a conformal method of passing through the singularity is specified for Dirac test fields. In § 3 our particle concept is stated explicitly. The exact Dirac solutions obtained in § 4 are the starting point for a Fock space formulation in § 5, which leads to a thermal spectrum for the created particles. In § 6 the development in time of the creation process which takes place near the Compton time is studied in detail. As shown in § 7 the equation of state for energy density and pressure of the created particles confirms the thermal spectrum. The appendix contains the essential facts of the WKB theory of Dirac particles in curved space–time. In the remaining part of the introduction the space–time is specified and a simple heuristic discussion anticipating the main results is given.

1.1. Cosmological model

As the cosmological background space–time by which particle creation is caused, we take the 3-flat Robertson–Walker geometry ($c=1$)

$$ds^2 = dt^2 - R^2(t)(dx^2 + dy^2 + dz^2) \tag{1.1a}$$

† For Dirac particles the same method leads to divergent results for energy density and pressure (Schäfer and Dehnen 1977).

or

$$ds^2 = R^2(\eta)(d\eta^2 - dx^2 - dy^2 - dz^2). \tag{1.1b}$$

As the expansion law we use that of a radiation-dominated universe, which proves to be a good description of the early stages of the universe, when particle creation is to be expected:

$$R = at^{1/2} = b\eta, \qquad b = a^2/2, \qquad a = \text{constant} > 0. \tag{1.2}$$

All statements refer to the observers moving along the preferred geodesics $x = y = z =$ constant (word lines of galaxies) and measuring t as proper time. They will be called cosmic observers.

This universe is characterised by having a 'big bang' and a singularity at $t = \eta = 0$ followed by decreasing velocity of expansion for $\eta > 0$ and an asymptotic passage to a universe with vanishing Hubble parameter for $\eta \to +\infty$. It is an essential point for the following considerations, that we complete this space–time to negative η passing through the curvature singularity at $\eta = 0$ in using the law $R(\eta)$ of (1.2) for $-\infty < \eta < 0$ as well. This means, we in fact use a space–time which has an asymptotically vanishing Hubble parameter for $\eta = -\infty$, contracts for $-\infty < \eta < 0$ towards a singularity at $\eta = 0$ and expands for $0 < \eta < +\infty$. This contraction–expansion law represents, apart from $\eta = 0$, for all η a solution of the Einstein equation for a radiation-filled universe.

Our justification for the completion of the universe by a sort of 'image universe' preceding the singularity is the following. There is no unambiguous way of introducing a particle interpretation in highly curved space–times because there are objections to a particle definition based on an instantaneous diagonalisation of the Hamiltonian. On the other hand, particle definition based on a WKB approximation needs a region of slow expansion or contraction (adiabatic region). To have such a region for the ingoing and outgoing particle states without changing the dynamics of the universe at early times (singularity) and without abandoning Einstein's equation, we need the expansion law (1.2) for negative values of η as well. Furthermore this is the limiting case of the universe with avoided singularity and a time-symmetric contraction–expansion law.

1.2. Heuristic discussion

Leaving aside details, a heuristic discussion of the essential facts of particle creation in Robertson–Walker universes leading to order of magnitude estimates can be given.

In a metric theory of gravitation in which gravitation is replaced by space–time curvature, the gravitational forces 'survive' as tidal forces between particles. These forces therefore should be responsible for the corresponding particle creation. The respective relative acceleration of two test particles is caused by the curvature of space–time. It is of the form (symbolically written equation of geodesic deviation) $\delta x/r^2$ where δx represents the infinitesimal distance of the particles and r^{-2} is given by the components of the Riemann tensor. For the universe (1.1) in question we have

$$1/r^2 \sim R'^2/R^2 \qquad \text{and} \qquad 1/r^2 \sim -R''/R \tag{1.3}$$

(differentiation with respect to t) where the greater expression has to be taken. We restrict to expansion laws

$$R = at^q, \qquad 0 < q \leqslant 1 \tag{1.4}$$

which imply

$$R'^2/R^2 = q^2/t^2, \qquad R''/R = (q^2 - q)/t^2. \tag{1.5}$$

In this geometry the test particles experience an acceleration towards each other.

Following Woodhouse (1977) we may say that a particle–antiparticle pair which is separated by a distance δx can gain from the tidal forces the energy $m(\delta x/r^2)\delta x$. If this energy reaches the order of the rest mass, the particles can become real

$$\frac{m(\delta x)^2}{r^2} \gtrsim m. \tag{1.6}$$

On the other hand, the distance δx over which virtual particles can spread is limited by the uncertainty relations $\Delta E\ \Delta t \sim 1$, ($\hbar = 1$). For the creation of a pair of particles an uncertainty $\Delta E \gtrsim m$ of the energy is needed which is only possible for a time $\Delta t \lesssim m^{-1}$. The virtual particles can therefore not last longer than m^{-1}, accordingly they can not propagate further than about m^{-1} and the extension of a cloud of virtual particles is about $x \lesssim \lambda_C = m^{-1}$ with Compton wavelength λ_C. Thus with (1.6): $r \lesssim \lambda_C$. For the expansion law (1.5) with $r \sim t$ we finally obtain $t \lesssim m^{-1} = t_C$. It is only before and up to the Compton time

$$t_C = \frac{\hbar}{mc^2} \approx 10^{-21}\ \mathrm{s} \tag{1.7}$$

(m is the electron mass) that a significant pair production occurs.

To take into account heuristically the production of particles which apart from their rest mass possess a *small* amount of kinetic energy, the right side of (1.6) is to be corrected

$$\frac{m(\delta x)^2}{r^2} \approx m + \alpha_1 \frac{p^2}{m}, \qquad \alpha_1 \approx 1. \tag{1.8}$$

Accordingly a 'quantum mechanical separation' of the virtual particles over more than a Compton wavelength is necessary. The pr bability for finding particles separated by a distance δx decays exponentially (Henley and Thirring 1962): $\exp(-mx)$. With (1.8) the probability for the creation of particles with momentum p is proportional to

$$\exp[-(mr + \alpha_2\, rp^2/m)] \qquad \text{with } \alpha_2 \approx 1.$$

In Robertson–Walker universes p diminishes according to $p = k/R$ with k = constant. The probability is therefore proportional to

$$\exp\left(-\alpha_2 \frac{rk^2}{R^2 m}\right). \tag{1.9}$$

The appearance of the exponential function in (1.9) suggests the conjecture that the spectrum of the created particles could be a non-relativistic thermal one. The necessary condition for this is that the argument of the exponential function is independent of time and proportional to k^2 because in this case it is of the approximate form $(-p^2/mT)$ with $p^2 \sim 1/R^2$ and temperature $T \sim R^2$.

Comparison of (1.9) with (1.3) and (1.5) shows that this is only the case for $q = \frac{1}{2}$. We therefore have that in the momentum spectrum of the created particles there appears an exponential function with an argument proportional to k^2 or p^2. But only in the case of the expansion law $R \sim t^{1/2}$ can this be taken as a hint that the particles are created with a thermal spectrum, which is then proportional to

$$\exp\left(-\alpha_3 \frac{k^2}{b^2 m}\right), \qquad \alpha_3 \approx 1. \tag{1.10}$$

The heuristic discussion above should not be overestimated. It suggests that some conjectures may be reasonable. Only the calculations in the following sections will finally confirm that for $R \sim t^{1/2}$ particles are created, that this mainly happens at about the Compton time and that their spectrum is a thermal one.

2. Dirac theory and conformal method

In this section we briefly review some elements of the generally covariant Dirac theory and specialise to our space–time where the domain of negative η needs additional considerations.

2.1. Dirac theory

With respect to an orthonormal tetrad field† $h^{\alpha}_{a}(x)$

$$h^{\alpha}_{a} h^{\beta}_{b} g_{\alpha\beta} = \eta_{ab} \tag{2.1}$$

the Dirac equation in curved space–time takes the form

$$i\gamma^{\mu}\Psi_{\|\mu} - m\Psi = 0 \tag{2.2}$$

with

$$\begin{aligned} &\Psi_{\|\mu} = \Psi_{|\mu} + \Gamma_{\mu}, \qquad \Gamma_{\mu} = \tfrac{1}{4} h^{\alpha}_{a\|\mu} h_{\alpha b} \gamma^{b} \gamma^{a} \\ &\gamma^{\mu} = h^{\mu}_{a} \gamma^{a}, \qquad \gamma^{(a}\gamma^{b)} = \eta^{ab}, \qquad \gamma^{(\mu}\gamma^{\nu)} = g^{\mu\nu}. \end{aligned} \tag{2.3}$$

The Dirac current j^{α} defined by means of $\bar{\Psi} = \Psi^{\dagger}\gamma^{(4)}$ is divergence-free:

$$j^{\alpha} = \bar{\Psi}\gamma^{\alpha}\Psi = \bar{\Psi} h^{\alpha}_{a} \gamma^{a} \Psi, \qquad j^{\alpha}{}_{\|\alpha} = 0. \tag{2.4}$$

This enables a hypersurface-independent normalisation on a hypersurface σ with time-like normal vector u^{α} using the integral (where $\mathrm{d}^3 V$ is the invariant volume element of σ)

$$\int_{\sigma} j^{\alpha} u_{\alpha} \, \mathrm{d}^3 V. \tag{2.5}$$

† $\hbar = 1$, $c = 1$. Signature of the metric tensor $g_{\alpha\beta}$: $(- - - +)$. $\|\alpha$ denotes the covariant and $|\alpha$ the partial derivative. $\alpha, \beta, \ldots = 1, \ldots, 4$ and $\hat{\alpha}, \hat{\beta}, \ldots = 1, 2, 3$ are tensor indices raised and lowered with $g_{\alpha\beta}$. $a, b, \ldots = 1, \ldots, 4$ and $\hat{a}, \hat{b}, \ldots = 1, 2, 3$, are tetrad indices raised and lowered with $\eta_{ab} = \mathrm{diag}(-1, -1, -1, +1)$. The corresponding geometrical object is a Riemannian scalar with regard to $a, b, \ldots$

$$M^{(1)} = M^{a=1}, \qquad M^{1} = M^{\alpha=1}, \qquad M_{(\alpha\beta)} = \tfrac{1}{2}(M_{\alpha\beta} + M_{\beta\alpha})$$

Standard γ^{a}-matrices as in Bjorken and Drell (1964)

$$\gamma^{\hat{a}} = \begin{bmatrix} 0 & \sigma^{\hat{a}} \\ \sigma^{\hat{a}} & 0 \end{bmatrix}, \qquad \gamma^{(4)} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The symmetric divergence-free energy–momentum tensor is given by

$$T_{\epsilon\mu} = \frac{i}{2}(\bar{\Psi}\gamma_{(\mu}\Psi_{\|\epsilon)} - \bar{\Psi}_{(\|\epsilon}\gamma_{\mu)}\Psi), \qquad T^{\epsilon\mu}{}_{\|\mu} = 0. \tag{2.6}$$

2.2. Conformal method

Our space–time with metric (1.1) is characterised by having a singularity at $t = \eta = 0$. As stated above our intention is, going backwards in time, to pass through the singularity into a space–time with metric (1.1*b*) where $\eta < 0$. Up to now there is no canonical method to do so with a quantum mechanical test field. We will introduce a procedure which could be characterised as a conformal method. It generalises considerations of Audretsch and Schäfer (1977).

For $\eta > 0$ our physical problem can be mapped into a completely equivalent problem which is formulated by means of a modified Dirac equation with regard to a conformally related metric. To do so we take the same manifold, the same coordinate system (x, y, z, η) as in (1.1*b*), refer to the same observer convergence ($x = \text{const}$, $y = \text{const}$, $z = \text{const}$) but introduce into the manifold instead of the metric $g_{\alpha\beta}$ of (1.1*b*) the conformally related Minkowski metric $\widetilde{g_{\alpha\beta}}$

$$g_{\alpha\beta} = R^2 \widetilde{g_{\alpha\beta}} \tag{2.7}$$

and correspondingly a tetrad field

$$\widetilde{h^{\alpha}_{a}} = R h^{\alpha}_{a}. \tag{2.8}$$

For a given congruence of world lines of particles in the underlying manifold describing a particle or charge flow, the corresponding currents are related according to (compare Audretsch and Schäfer 1977)

$$\widetilde{j^{\alpha}} = R^4 j^{\alpha}. \tag{2.9}$$

Because of (2.4), (2.7) and (2.8) this implies that Dirac fields in the two space–times are to be connected by

$$\Phi = R^{3/2}\tilde{\Psi}, \qquad \Phi^{\dagger} = R^{3/2}\Psi^{\dagger} \tag{2.10}$$

where we have written Φ instead of $\tilde{\Psi}$. Ψ is the solution of the Dirac equation (2.2) evaluated with $g_{\alpha\beta}$ and h^{α}_{a}. Adjusting the tetrads along the coordinate lines

$$h^{\alpha}_{a} = \frac{1}{R}\delta^{\alpha}_{a}, \qquad \widetilde{h^{\alpha}_{a}} = \delta^{\alpha}_{a} \tag{2.11}$$

this Dirac equation takes the form

$$i\delta^{\alpha}_{a}\gamma^{a}\Psi_{|\alpha} + i\tfrac{3}{2}(\overset{0}{R}/R)\gamma^{4}\Psi - Rm\Psi = 0. \tag{2.12}$$

The corresponding equation for Φ is, because of (2.10),

$$i\delta^{\alpha}_{a}\gamma^{a}\Phi_{|\alpha} - Rm\Phi = 0. \tag{2.13}$$

With the tangent vector to the observer world lines

$$u^{\alpha} = \frac{1}{A}\delta^{\alpha}_{4}, \qquad \tilde{u}^{\alpha} = \delta^{\alpha}_{4}, \qquad A = R \tag{2.14}$$

the normalisation with respect to the $\eta = \text{const}$ hypersurface takes the form

$$\int_\sigma j^\alpha u_\alpha \, d^3V = \int_\sigma \widetilde{j^\alpha} \tilde{u}_\alpha \, d^3\widetilde{V} = \int_\sigma \Phi^\dagger \Phi \, dx \, dy \, dz = 1 \tag{2.15}$$

where we have used

$$d^3V = A^3 \, d^3\widetilde{V} = A^3 \, dx \, dy \, dz, \qquad A = R. \tag{2.16}$$

This implies for the scalar product of the Φ:

$$\langle \Phi_1, \Phi_2 \rangle = \int_\sigma \Phi_1^\dagger \Phi_2 \, dx \, dy \, dz. \tag{2.17}$$

After some calculation starting from (2.6) and using (2.11), (2.10), (2.14) and (2.16) we obtain for the energy as seen by the cosmic observer u^α

$$T_{\alpha\beta} u^\alpha u^\beta \, d^3V = \frac{A}{R^2} \frac{i}{2} (\Phi^\dagger \Phi_{|4} + \Phi^\dagger_{|4} \Phi) \, dx \, dy \, dz = \frac{A}{2R^2} (\Phi^\dagger \tilde{H} \Phi - (\tilde{H}\Phi)^\dagger \Phi) \, dx \, dy \, dz \tag{2.18}$$

with

$$\tilde{H}\Phi = \frac{1}{i} \gamma^{(4)} \gamma^{\hat{a}} \delta^{\hat{a}}_{\hat{a}} \Phi_{|\hat{a}} + Rm\gamma^{(4)} \Phi. \tag{2.19}$$

Summarising, we may say that for $\eta > 0$ our physical problem which refers to the space–time with metric $g_{\alpha\beta}$ and which is characterised by $T^{\alpha\beta}$, j^α, u^α and by the field equation (2.12) for Ψ is mapped into a completely equivalent problem which refers to the space–time with metric $\widetilde{g_{\alpha\beta}}$ (Minkowski space–time) and which is characterised by $\tilde{H}, \widetilde{j^\alpha}$, $\widetilde{u^\alpha}$ and by the field equations (2.13) for the field Φ containing a time-dependent potential. Note that for the latter Minkowski space–time problem there is no singular behaviour at the hypersurface $\eta = 0$ where $R(\eta = 0) = 0$. So we may continue the Dirac equation analytically into the domain $\eta \leq 0$ in using all expressions with tilde and the law $R = b\eta$ of (1.2) everywhere. This then defines the equivalent physical problem for all values of η. When going back for $\eta < 0$ to the original problem, one has to use $A = |\eta|$ in order to have future-pointing observer velocity in (2.14) and a positive volume element in (2.16).

In a strict sense a 'passage through the singularity' is impossible in the framework of classical general relativity. Singular points are not a part of the manifold. To establish nevertheless for test fields and test particles a connection with the world before the big bang, an additional procedure generalising the theory is needed. This will necessarily show features which are *ad hoc*, although the procedure itself may seem natural. In the remainder of this section we want to discuss these features more precisely. The guiding physical idea is that the singularity at $\eta = 0$ will not occur in a more realistic universe and that in this case for $\eta < 0$ the universe will be completed by a mirror-like universe. The occurrence of a singularity at $\eta = 0$ is then the limiting case of this situation. Its study is the subject of this paper. It is of importance especially if one wants to answer the question in which way the singularity itself is responsible for the particle creation. In the following it is to be kept in mind that we are discussing quantum mechanics in a given, i.e. otherwise determined, space–time by means of test fields. It is only the physics of these test fields which is extended beyond $\eta = 0$. The features with a certain *ad hoc* character are:

(i) We join two identical space–times $(\mathcal{M}_1, g_1)$ and $(\mathcal{M}_2, g_2)$ with metrics g_1 and g_2 mirror-like together. $(\mathcal{M}_2, g_2)$ is thereby the Robertson–Walker universe in question.

(ii) The two space–times have a scalar singularity at $t = \eta = 0$ (for further details see Hawking and Ellis (1973) and Ellis and Schmidt (1977)). It is a matter singularity, i.e. it is caused by the Ricci tensor, and not a conformal singularity. This is related to the fact that the singularity can be avoided for a presumably more physical matter content of the universe. Because the singularity is no conformal singularity, it is posible to go over to the conformally related space–times $(\mathcal{M}_1, \widetilde{g_1})$ and $(\mathcal{M}_2, \widetilde{g_2})$ which may have a regular boundary at the place of the scalar singularity. In our case $(\mathcal{M}_1, \widetilde{g_1})$ and $(\mathcal{M}_2, \widetilde{g_2})$ both are a half Minkowski space–time. It is important now that it is possible to formulate in $(\mathcal{M}_i, \widetilde{g_i})$ $(i = 1, 2)$ by means of correspondingly altered field equations for the Dirac test field a physical problem which is completely equivalent to the original one in $(\mathcal{M}_i, g_i)$. Furthermore, because the boundary points of the $(\mathcal{M}_i, \widetilde{g_i})$ are regular, it is mathematically and physically trivial to join them together. With regard to the equivalent physical problem for the Diract test field, the only new fact then is that now initial conditions specified in $(\mathcal{M}_1, \widetilde{g_1})$ or $(\mathcal{M}_1, g_1)$ will influence the physics of the Dirac field in $(\mathcal{M}_2, \widetilde{g_2})$ and therefore $(\mathcal{M}_2, g_2)$.

(iii) Finally, because the theory of spin-$\frac{1}{2}$ fields is formulated by means of a tetrad field instead of the metric only, it needs a further specification. In our case, the concept 'mirror-like' of (i) for $\eta < 0$ could as well be realised by using $|R|$ instead of $R < 0$. We choose the latter to obtain an analytical behaviour for equation (2.13).

The point (ii) characterises the conformal method itself. The other points contain additional reasonable assumptions. We mention that the whole procedure above could as well be interpreted in a restricting sense as a method defining initial conditions for the spin-$\frac{1}{2}$ field in the space–time $(\mathcal{M}_2, g_2)$ only.

In a subsequent paper we will treat the creation of Klein–Gordon particles in a time-symmetric universe without singularity which approaches for $\eta > 0$ the space–time $(\mathcal{M}_2, g_2)$ of this paper as a limiting case in the sense discussed above.

3. Particle concept

For the definition of particles we will use a generally covariant approach, which is based on solutions of the WKB equation in the respective space–time.

Definition. If at a time η_0 (or asymptotically) a system of WKB solutions is (i) complete with respect to the Dirac norm and (ii) fulfills the dynamical field equation, i.e. is also a solution of the Dirac equation, it describes at the respective point of time η_0 (or asymptotically) particles.

We mention that by (ii) the usually required condition that 'the Dirac solutions should approach WKB solutions' in order that a particle interpretation is possible, is specified in a rather *strict* sense because a demand concerning the derivative is included. Condition (ii) is necessary because only the Dirac equation describes Dirac particles. According to the definition above a particle interpretation can only be introduced for times η_0 when for all modes no particle creation or annihilation takes place. For a Dirac solution

$$\Psi^{\text{Dirac}}(x) = \sum_{\boldsymbol{k},s} (c_{\boldsymbol{k},s}(\eta)\Psi^{\text{WKB}}_{+1,\boldsymbol{k},s}(x) + d^*_{\boldsymbol{k},s}(\eta)\Psi^{\text{WKB}}_{-1,-\boldsymbol{k},-s}(x)) \tag{3.1}$$

condition (ii) implies at η_0

$$\frac{\partial}{\partial\eta} c_{k,s}\big|_{\eta_0} = \frac{\partial}{\partial\eta} d_{k,s}\big|_{\eta_0} = 0. \tag{3.2}$$

As shown in the appendix in our space–time a particle interpretation and correspondingly the introduction of a vacuum state is only possible at $\eta = \pm\infty$. Condition (ii) is not fulfilled at $\eta = 0$.

4. Exact solutions

From the Dirac equation (2.3)

$$(\mathrm{i}\gamma^a \partial_\alpha - mR(\eta))\Phi = 0, \qquad \partial_\alpha = \{\partial_x, \partial_y, \partial_z, \partial_\eta\} \tag{4.1}$$

we go over by the substitution

$$\Phi = (-\mathrm{i}\gamma^a \partial_a - Rm)\phi \tag{4.2}$$

to the squared equation

$$[-\eta^{ab}\partial_a\partial_b + \mathrm{i}\gamma^{(4)}\dot{R}m - R^2m^2]\phi = 0 \tag{4.3}$$

where the dot denotes differentiation with respect to η. We solve (4.3) with the *ansatz*

$$\phi = f(\eta)\Gamma \exp(-\mathrm{i}k_{\hat{a}}x^{\hat{a}}) \tag{4.4}$$

where the constant spinor Γ is defined by

$$\gamma^{(4)}\Gamma = \epsilon\Gamma, \qquad \epsilon = \pm 1. \tag{4.5}$$

For the standard representation of the γ^a (Bjorken and Drell 1964) Γ becomes

$$\Gamma_1^{\epsilon=1} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \qquad \Gamma_{-1}^{\epsilon=1} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \qquad \Gamma_1^{\epsilon=-1} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \qquad \Gamma_{-1}^{\epsilon=-1} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \tag{4.6}$$

and (4.3) reduces to a differential equation for $f(\eta)$:

$$\ddot{f} + (R^2m^2 - \mathrm{i}\epsilon m\dot{R} + k^2)f = 0. \tag{4.7}$$

If we now specialise to the expansion law $R = b\eta$ of the radiation-filled universe we obtain the differential equation for parabolic cylinder functions:

$$\frac{\mathrm{d}^2 f}{\mathrm{d}z^2} + (\nu + \tfrac{1}{2} - \tfrac{1}{4}z^2)f = 0 \tag{4.8}$$

where

$$\nu = \tfrac{1}{2}[-+\epsilon -+(\mathrm{i}k^2/bm) - 1] \tag{4.9}$$

$$z = \pm(1 + - \mathrm{i})\surd(bm)\eta \tag{4.10}$$

(compare Magnus *et al* 1966)†. For $\epsilon=+1$ solutions are given by

$$f(\eta)=D_{\mathrm{i}k^2/2bm}[\pm\mathrm{i}(1-\mathrm{i})\sqrt{(bm)}\eta] \tag{4.11a}$$

$$f(\eta)=D_{(-\mathrm{i}k^2/2bm)-1}[\pm\mathrm{i}(-1+\mathrm{i})\sqrt{(bm)}\eta]. \tag{4.11b}$$

For a fundamental system it is sufficient to restrict to the upper sign in (4.11).

To obtain Φ we have to return to (4.2). When applying the corresponding operator to (4.11*a*) and (4.11*b*) we make use of

$$\frac{\partial}{\partial z}D_\nu(z)=-\tfrac{1}{2}zD_\nu(z)+\nu D_{\nu-1}(z) \tag{4.12}$$

and

$$\frac{\partial}{\partial z}D_\nu(z)=\tfrac{1}{2}zD_\nu(z)-D_{\nu+1}(z) \tag{4.13}$$

respectively. After normalisation this finally leads to the solutions

$$\begin{aligned}{}^{-}_{+}\Phi_{\boldsymbol{k},s}(x)=\exp\left(-\frac{\pi}{8}\frac{k^2}{bm}\right)[(2\pi)^3k^2]^{-1/2}\exp(\pm\mathrm{i}k_{\hat{a}}x^{\hat{a}})\{u_sD_{\mathrm{i}k^2/2bm}[\pm(1-\mathrm{i})\sqrt{(bm)}\eta]\\ +(1-\mathrm{i})\sqrt{(bm)}\frac{k^2}{2bm}\hat{u}_sD_{(\mathrm{i}k^2/2bm)-1}[\pm(1-\mathrm{i})\sqrt{(bm)}\eta]\}\end{aligned} \tag{4.14a}$$

and

$$\begin{aligned}{}^{+}_{-}\Phi_{\boldsymbol{k},s}=\exp\left(-\frac{\pi}{8}\frac{k^2}{bm}\right)[2(2\pi)^3bm]^{-1/2}\exp(\mp\mathrm{i}k_{\hat{a}}x^{\hat{a}})\{u_sD_{-(\mathrm{i}k^2/2bm)-1}[\mp(1+\mathrm{i})\sqrt{(bm)}\eta]\\ +(1-\mathrm{i})\sqrt{(bm)}\hat{u}_sD_{-\mathrm{i}k^2/2bm}[\mp(1+\mathrm{i})\sqrt{(bm)}\eta]\}\end{aligned} \tag{4.14b}$$

with spin quantum number $s=\pm1$ and

$$\hat{u}_1=\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\qquad \hat{u}_{-1}=\begin{pmatrix}0\\1\\0\\0\end{pmatrix} \tag{4.15a}$$

$$u_1=\begin{pmatrix}0\\0\\-k_3\\-k_1-\mathrm{i}k_2\end{pmatrix},\qquad u_{-1}=\begin{pmatrix}0\\0\\-k_1+\mathrm{i}k_2\\k_3\end{pmatrix}. \tag{4.15b}$$

Thereby the (upper, lower) sign at Φ corresponds to the (upper, lower) sign at the right side of the equation. The upper signs in (4.14*a*) and (4.14*b*) represent together for all η a complete orthonormal system. the same is true for the lower sign. For $\epsilon=-1$ the same result follows; we can therefore restrict to $\epsilon=+1$.

An interpretation of the solutions above is based on the considerations in the appendix concerning WKB solutions: the action S is of the form

$$S=-\frac{bm}{2}\left\{\eta\left(\eta^2+\frac{k^2}{b^2m^2}\right)^{1/2}+\frac{k^2}{b^2m^2}\ln\left[\eta+\left(\eta^2+\frac{k^2}{b^2m^2}\right)^{1/2}\right]\right\}-k_{\hat{a}}x^{\hat{a}} \tag{4.16}$$

† To indicate that the two choices of the signs are independent we write (±) and (+ −).

and shows for $|\eta|\to\infty$ the asymptotic behaviour

$$S\to -\frac{bm}{2}\eta|\eta|-k_{\hat{a}}x^{\hat{a}}. \tag{4.17}$$

Therefore with the energy–momentum tensor (A.15) we have asymptotically the following correspondence in the sense of § 3:

$$\begin{aligned}\Phi &\simeq e^{iS} \leftrightarrow \text{positive energy} \leftrightarrow \text{particle}\\ \Phi &\simeq e^{-iS} \leftrightarrow \text{negative energy} \leftrightarrow \text{antiparticle}.\end{aligned} \tag{4.18}$$

On the other hand, the asymptotic behaviour of the parabolic cylinder functions for $|z|\to\infty$ with finite index ν

$$D_\nu(z)\cong e^{-z^2/4}z^{\nu} \qquad \text{for } |\arg z|<3\pi/4 \tag{4.19a}$$

$$D_\nu(z)\cong e^{-z^2/4}z^{\nu}-\frac{\sqrt{(2\pi)}}{\Gamma(-\nu)}e^{-i\pi\nu}e^{+z^2/4}z^{-\nu-1} \qquad \text{for } \pi/4<\arg z<5\pi/4 \tag{4.19b}$$

implies according to (4.19a) and (4.14) the interpretation

$$\left.\begin{matrix}{}_+\Phi_{\boldsymbol{k}}\\ {}^+\Phi_{\boldsymbol{k}}\end{matrix}\right\}\text{particle with momentum } +\boldsymbol{k} \text{ at}\begin{cases}\eta\to-\infty\\ \eta\to+\infty\end{cases}$$

$$\left.\begin{matrix}{}_-\Phi_{\boldsymbol{k}}\\ {}^-\Phi_{\boldsymbol{k}}\end{matrix}\right\}\text{antiparticle with momentum } -\boldsymbol{k} \text{ at}\begin{cases}\eta\to-\infty\\ \eta\to+\infty\end{cases} \tag{4.20}$$

where on the right side of (4.14) the upper or lower sign is correspondingly chosen. Because of (4.19b) the remaining choices for the sign do not correspond to an asymptotic WKB behaviour.

5. Thermal distribution of created particles

To determine the momentum spectrum of possibly created particles we go over to quantum field theory in a Fock space formulation by introducing particle creation and annihilation operators. We decompose the now quantised field Φ according to

$$\Phi(x)=\sum_{\boldsymbol{k},s}[{}_+\Phi_{\boldsymbol{k},-s}(x)a^{\text{in}}_{\boldsymbol{k},s}+{}_-\Phi_{-\boldsymbol{k},-s}(x)(b^{\text{in}}_{\boldsymbol{k},s})^{\dagger}] \tag{5.1a}$$

$$\Phi(x)=\sum_{\boldsymbol{k},s}[{}^+\Phi_{\boldsymbol{k},s}(x)a^{\text{out}}_{\boldsymbol{k},s}+{}^-\Phi_{-\boldsymbol{k},-s}(x)(b^{\text{out}}_{\boldsymbol{k},s})^{\dagger}] \tag{5.1b}$$

where the operators obey the anticommutation relations

$$[a^{\text{in}}_{\boldsymbol{k},s},a^{\text{in}\dagger}_{\boldsymbol{k}',s'}]_+=[b^{\text{in}}_{\boldsymbol{k},s},b^{\text{in}\dagger}_{\boldsymbol{k}',s'}]_+=\delta(\boldsymbol{k}-\boldsymbol{k}')\delta_{s,s'} \tag{5.2}$$

and the in-vacuum is defined at $\eta=-\infty$ by

$$a^{\text{in}}_{\boldsymbol{k},s}|{}^{\text{in}}_{\text{vac}}\rangle=b^{\text{in}}_{\boldsymbol{k},s}|{}^{\text{in}}_{\text{vac}}\rangle=0 \tag{5.3a}$$

$$\langle{}^{\text{in}}_{\text{vac}}|{}^{\text{in}}_{\text{vac}}\rangle=1. \tag{5.3b}$$

The corresponding relations apply for out-operators and the out-vacuum at $\eta=+\infty$ (the other anticommutators vanish). The mean value $\bar{N}_{\boldsymbol{k},s}$ of the out-particles of the mode $(\boldsymbol{k}, s)$ wich are generated out of the in-vacuum is given by

$$\bar{N}_{\boldsymbol{k},s} = \langle {}^{\text{in}}_{\text{vac}} | (a^{\text{out}}_{\boldsymbol{k},s})^\dagger a^{\text{out}}_{\boldsymbol{k},s} | {}^{\text{in}}_{\text{vac}} \rangle. \tag{5.4}$$

Using (5.2) and the orthonormality relation of the $\Psi_{\boldsymbol{k},s}$ we obtain from (5.1)

$$a^{\text{out}}_{\boldsymbol{k},s} = (2\pi)^3 \, {}^{+}\hat{\Phi}^\dagger_{\boldsymbol{k},s} \, {}_{+}\hat{\Phi}_{\boldsymbol{k},s} a^{\text{in}}_{\boldsymbol{k},s} + (2\pi)^3 \, {}^{+}\hat{\Phi}^\dagger_{\boldsymbol{k},s} \, {}_{-}\hat{\Phi}_{\boldsymbol{k},s} (b^{\text{in}}_{-\boldsymbol{k},-s})^\dagger \tag{5.5}$$

$$(b^{\text{out}}_{-\boldsymbol{k},-s})^\dagger = (2\pi)^3 \, {}^{-}\hat{\Phi}^\dagger_{\boldsymbol{k},s} \, {}_{+}\hat{\Phi}_{\boldsymbol{k},s} a^{\text{in}}_{\boldsymbol{k},s} + (2\pi)^3 \, {}^{-}\hat{\Phi}^\dagger_{\boldsymbol{k},s} \, {}_{-}\hat{\Phi}_{\boldsymbol{k},s} (b^{\text{in}}_{-\boldsymbol{k},-s})^\dagger \tag{5.6}$$

with

$${}^{-}_{+}\Phi_{\boldsymbol{k},s}(x) = \exp(\pm i k_{\hat{a}} x^{\hat{a}}) \, {}^{-}_{+}\hat{\Phi}_{\boldsymbol{k},s}(\eta). \tag{5.7}$$

Because of (4.14) we have

$$(2\pi)^3 \, {}^{+}\hat{\Phi}^\dagger_{\boldsymbol{k},s} \, {}_{+}\hat{\Phi}_{\boldsymbol{k},s} = -i / M^*_{k^2,s} \tag{5.8}$$

and correspondingly

$$(2\pi)^3 \, {}^{-}\hat{\Phi}^\dagger_{\boldsymbol{k},s} \, {}_{-}\hat{\Phi}_{\boldsymbol{k},s} = i / M_{k^2,s} \tag{5.9}$$

where $M_{k^2,s}$ is given by

$$M_{k^2,s} = -\frac{1}{2} \Gamma\left(\frac{ik^2}{2mb}\right) \cdot \left(\frac{k^2}{\pi bm}\right)^{1/2} \exp\left(\frac{\pi k^2}{4bm}\right). \tag{5.10}$$

Furthermore with

$$|M_{k^2,s}|^2 = [1 - \exp(-\pi k^2/bm)]^{-1} \tag{5.11}$$

and using (5.2), (5.5) and (5.6) we find

$$|(2\pi)^3 \, {}^{+}\hat{\Phi}^\dagger_{\boldsymbol{k},s} \, {}_{-}\hat{\Phi}_{\boldsymbol{k},s}|^2 = \exp(-\pi k^2/bm) \tag{5.12}$$

$$|(2\pi)^3 \, {}^{-}\Phi^\dagger_{\boldsymbol{k},s} \, {}_{+}\hat{\Phi}_{\boldsymbol{k},s}|^2 = \exp(-\pi k^2/bm). \tag{5.13}$$

This finally implies with (5.4):

$$\bar{N}_{\boldsymbol{k},s} = \exp(-\pi k^2/bm) \delta^3(\boldsymbol{k} - \boldsymbol{k}). \tag{5.14}$$

The same result can be obtained for antiparticles, so that the final expression for the total number of particles and antiparticles together created in the mode $(\boldsymbol{k}, s)$ is given by

$$\bar{N}^{\text{total}}_{\boldsymbol{k},s} = 2 \exp(-\pi k^2/bm) \delta^3(\boldsymbol{k} - \boldsymbol{k}) \tag{5.15}$$

Additionally, using the 'golden rule'

$$\delta^{(3)}(\boldsymbol{k} - \boldsymbol{k}) = \frac{1}{(2\pi)^3} \int d^3x = \frac{\Sigma}{(2\pi)^3} \tag{5.16}$$

and summing up spin indices we obtain as total number of particles generated per unit coordinate volume with momentum out of the interval $[\boldsymbol{k}, \boldsymbol{k} + d\boldsymbol{k}]$†

$$\frac{\bar{N}^{\text{total}}_{\boldsymbol{k}}}{\Sigma} d^3\boldsymbol{k} = 4 \exp\left(-\frac{\pi k^2}{bm}\right) \frac{1}{(2\pi)^3} d^3k. \tag{5.17}$$

† The quantum mechanical probability interpretation which is based on repeated measurements does not contradict the fact that there is only one universe and one cosmic evolution, because these repeated measurements can be done in different finite volumes.

Referring to the volume as measured by the cosmic observer, the total number of particles $\bar{N}^{\text{total}}$ which are created per unit volume element with arbitrary momentum and arbitrary direction of spin is given by ($V = R^3\Sigma$)

$$\frac{\bar{N}^{\text{total}}}{V} = \int_k \frac{\bar{N}_k^{\text{total}}}{V} \mathrm{d}^3k = \frac{(bm)^{3/2}}{2\pi^3 R^3}. \tag{5.18}$$

It follows from (5.14) that the particles are created with a non-relativistic thermal spectrum with vanishing chemical potential (Maxwell–Boltzmann distribution). Note that all considerations above are concerned with asymptotic out-states. Because of the time dependence $p(t) = k/R(t)$ of the momentum the particles are then non-relativistic, and the argument of the exponential function in (5.14) represents the quotient of kinetic energy and temperature T both as measured by the cosmic observer:

$$T = b/2\pi R^2 k_{\mathrm{B}} \tag{5.19}$$

where k_{B} is Boltzmann's constant. To obtain (5.19) one has to use the single-particle energy $p^2/2m$ instead of the kinetic energy of the pair, because the gas of particles and antiparticles does not react in later thermodynamical interactions in finite volumes as if it is composed out of correlated pairs. With (1.2) and introducing

$$T_{\mathrm{C}} = \frac{mc^2}{k_{\mathrm{B}}} \approx 0{\cdot}6 \times 10^9\ \mathrm{K} \tag{5.20}$$

which may be called the Compton temperature, the particle temperature T takes the form

$$T = \frac{1}{4\pi} \frac{t_{\mathrm{C}}}{t} T_{\mathrm{C}} = \frac{1}{2\pi} H t_{\mathrm{C}} T_{\mathrm{C}} \tag{5.21}$$

where $H = \dot{R}/R$ is the Hubble parameter.

Because the chemical potential of the non-relativistic thermal particle distribution vanishes, the entropy density per coordinate volume is given by

$$S/\Sigma = 4k_{\mathrm{B}}\tfrac{5}{2}(2\pi/mk_{\mathrm{B}}T)^{-3/2} \tag{5.22}$$

which is according to (5.19)

$$S/\Sigma = \tfrac{5}{2}k_{\mathrm{B}}\bar{N}^{\text{total}}/\Sigma. \tag{5.23}$$

It remains constant during the expansion of the universe.

For the case of cosmological models with an event horizon, it has recently been shown by Gibbons and Hawking (1977) that an observer on a geodesic will find a background of thermal radiation coming from the event horizon. Because there is no event horizon in the space–time which we are discussing, the thermal radiation deduced above cannot be explained by the mechanism used by Gibbons and Hawking (1977).

We mention without stating the details of the calculation that the result (5.17) with a factor two instead of four is also obtained, when the same problem is calculated in the framework of a Klein–Gordon theory if the passage through $\eta = 0$ is performed analytically as in § 2.

6. Development in time of the creation process

We are treating the process of particle creation by means of an 'in-out formalism' which leads to rigorous results only in the asymptotic region of time-like infinity $t \to \infty$. Nevertheless a detailed analysis showing when the exact Dirac solutions of (4.14) pass over to WKB solutions allows additional statements concerning the time when the respective modes can be interpreted as already created particles.

The asymptotic expansion

$$D_\nu(z) \approx e^{-z^2/4} z^\nu \left(1 - \frac{\nu(\nu-1)}{2z^2} + \frac{\nu(\nu-1)(\nu-2)(\nu-3)}{2\,.\,4\,.\,z^4} \mp \cdots\right), \qquad |\arg z| < \frac{3\pi}{4} \tag{6.1}$$

is valid for the region

$$|z| \gg 1 \tag{6.2a}$$

$$|z| \gg |\nu|. \tag{6.2b}$$

It shows that the generation process has finished, i.e. that the respective particles are present, if the relations corresponding to (6.2) are fulfilled. For our case we have

$$|z| \gg 1 \leftrightarrow \eta \gg (2bm)^{-1/2} \leftrightarrow t \gg 1/4m = \tfrac{1}{4} t_C \tag{6.3a}$$

$$|z| \gg |\nu| \leftrightarrow \eta \gg k^2/(2bm)^{3/2} \leftrightarrow t \gg k^4/4m^3a^4. \tag{6.3b}$$

According to (6.3a) it is at about the Compton time t_C and afterwards that the cosmic expansion has led to the creation of particles. Condition (6.3b) is also fulfilled for all $k \leq k' = \sqrt{(2bm)} = a\sqrt{m}$. This means that at a time t which fulfills (6.3a), all modes with $k \leq k'$ have already become particles. Using (5.17) it follows from

$$\int_0^{k'} \frac{2}{\pi^2} \exp\left(-\frac{\pi k^2}{bm}\right) k^2 \, dk$$

$$= \frac{1}{2\pi^3} (bm)^{3/2} \left\{ \operatorname{erf}\left[\left(\frac{\pi}{bm}\right)^{1/2} k'\right] - \frac{2k'}{\sqrt{(bm)}} \exp\left(-\frac{\pi k'^2}{bm}\right)\right\} \tag{6.4}$$

(where erf denotes the error function) that already 99% of the total number of particles are generated when equation (6.3a) is fulfilled. We may therefore conclude that 99% of the particles created all together are created at about the Compton time. Present day theories about the early stages of the universe lead to a radiation-dominated universe up to $t \approx 10^4$ years. Our choice of the expansion law $R \sim t^{1/2}$ (radiation-filled universe) for the study of particle creation seems therefore to be justified.

There remains the question if the particles are created with a relativistic momentum $p = k/R$. For $k \leq k'$ we have

$$p^2/m^2 = k^2/m^2a^2t \leq t_C/t. \tag{6.5}$$

For these 99% of the particles we may, according to (6.3a) at least, conclude that they are not created ultra-relativistically. The same applies for the remaining 1% which are created with high values of k but because of (6.3b) at later times t. This result is plausible because our field equation becomes conformally invariant for $m \to 0$.

We compare at Compton time $t = t_C$ the characteristic properties of the gas of created particles with the radiation content of the universe by using Einstein's equation. Assuming that the particles are created with electron rest mass m, the respective energy density ρ obtained from (5.18) and the energy density of the cosmic radiation ρ_r are related by

$$\left.\frac{\rho}{\rho_r}\right|_{t=t_C} = \frac{4\sqrt{2}}{3\pi^2}\frac{Gm^2}{\hbar c} \approx 10^{-46}. \tag{6.6}$$

From this result it seems reasonable to neglect the back reaction of the created particles on the metric during the creation. The particle temperature T of (5.21) at Compton time is related to the temperature of the radiation content of the universe according to

$$\left.\frac{T}{T_r}\right|_{t=t_C} = \frac{1}{2\pi}\left(\frac{2\pi^3}{45}\right)^{1/4}\left(\frac{Gm^2}{\hbar c}\right)^{1/4} \approx 10^{-12}. \tag{6.7}$$

Because of (5.18) the number of particles created per Compton volume λ_C^3 is at Compton time t_C

$$\frac{\bar{N}^{\text{total}}}{V}\frac{1}{m^3} = \frac{1}{2^{5/2}\pi^2} \approx \frac{1}{175}. \tag{6.8}$$

7. Energy density and pressure of the created particles

In the following we discuss energy density and pressure of the created particles. For the space–time of § 1 using the convention of § 2 the energy–momentum tensor $T^{\alpha\beta}$ of the Dirac field is given (we omit details of the calculation, $\Psi = R^{-3/2}\Phi$) by

$$T_{\alpha\beta} = \frac{\mathrm{i}}{2}\frac{1}{R^2}(\bar{\Phi}\gamma_a\delta^a_{(\alpha}\Phi_{|\beta)} - \bar{\Phi}_{(|\alpha}\delta^b_{\beta)}\gamma_b\Phi) \tag{7.1}$$

We pass over the asymptotic behaviour ($\eta \to \infty$) of the energy–momentum tensor in second quantisation by

$$\langle{}^{\text{in}}_{\text{vac}}| : T^{\text{out}}_{\alpha\beta} : |{}^{\text{in}}_{\text{vac}}\rangle = \Theta_{\alpha\beta} \tag{7.2}$$

where $:T^{\text{out}}_{\alpha\beta}:$ is obtained by inserting (5.1*b*) into (7.1) and normal ordering with regard to the out-operators.

By explicit calculations, again we omit the details, it can be shown that the off-diagonal elements of $\Theta_{\alpha\beta}$ vanish and that

$$\Theta_1{}^1 = \Theta_2{}^2 = \Theta_3{}^3. \tag{7.3}$$

This is plausible because of the symmetry of the space–time and the corresponding symmetry of the vacuum. The component $\Theta_4{}^4$ is the energy density and $-\Theta_3{}^3$ is the isotropic pressure both as measured by the cosmic observer relative to the measured volume. In contrast to this, measured energy $\hat{\rho}$ and pressure $\hat{p}$ taken per coordinate volume are given by

$$\hat{\rho} = R^3\Theta_4{}^4 \tag{7.4}$$

$$\hat{p} = -R^3\Theta_3{}^3. \tag{7.5}$$

Using the asymptotic behaviour of (6.1) the quantities $\hat{\rho}$ and $\hat{p}$ explicitly take the form of series with respect to inverse powers of R:

$$\rho = \hat{\rho}_0 + R^{-1}\hat{\rho}_{-1} + R^{-2}\hat{\rho}_{-2} + \ldots \tag{7.6}$$

$$\hat{p} = R^{-1}\hat{p}_{-1} + R^{-2}\hat{p}_{-2} + \ldots . \tag{7.7}$$

The terms proportional to even powers of R^{-1} come from the diagonal terms of $:T^{\text{out}}_{\alpha\beta}:$, the terms proportional to odd powers of R^{-1} result from off-diagonal terms of $:T^{\text{out}}_{\alpha\beta}:$. For $\hat{\rho}_0$ we obtain

$$\hat{\rho}_0 = m \sum_s \int_k \mathrm{d}^3k \left(\frac{\langle {}^{\text{in}}_{\text{vac}} | (a^{\text{out}}_{k,s})^\dagger a^{\text{out}}_{k,s} | {}^{\text{in}}_{\text{vac}} \rangle}{(2\pi)^3 \delta^3(k-k)} + \frac{\langle {}^{\text{in}}_{\text{vac}} | (b^{\text{out}}_{k,s})^\dagger b^{\text{out}}_{k,s} | {}^{\text{in}}_{\text{vac}} \rangle}{(2\pi)^3 \delta^3(k-k)} \right). \tag{7.8}$$

Applying (5.4) and (5.14) we obtain

$$\hat{\rho}_0 = 4 \int_k \frac{\mathrm{d}^3k}{(2\pi)^3} m \exp\left(-\frac{\pi k^2}{bm}\right) = \frac{m}{2\pi^3} (bm)^{3/2}. \tag{7.9}$$

Correspondingly we find

$$\hat{\rho}_{-2} = \sum_s \int_k \mathrm{d}^3k \frac{k^2}{2m} \left(\frac{\langle {}^{\text{in}}_{\text{vac}} | (a^{\text{out}}_{k,s})^\dagger a^{\text{out}}_{k,s} | {}^{\text{in}}_{\text{vac}} \rangle}{(2\pi)^3 \delta(k-k)} + \frac{\langle {}^{\text{in}}_{\text{vac}} | (b^{\text{out}}_{k,s})^\dagger b^{\text{out}}_{k,s} | {}^{\text{in}}_{\text{vac}} \rangle}{(2\pi)^3 \delta(k-k)} \right)$$

$$= 4 \int_k \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{k^2}{2m} \exp\left(-\frac{\pi k^2}{bm}\right) = \frac{3}{8} \frac{1}{\pi^4} \frac{1}{m} (bm)^{5/2} \tag{7.10}$$

$$\hat{p}_{-2} = 4 \sum_s \int_k \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{k_3^2}{m} \exp\left(-\frac{\pi k^2}{bm}\right). \tag{7.11}$$

Comparison of (7.11) with (7.10) leads to the following relation:

$$\hat{p}_{-2} = \tfrac{2}{3}\hat{\rho}_{-2}. \tag{7.12}$$

The remaining terms are of the form

$$\hat{\rho}_{-1} = \begin{Bmatrix} \text{terms containing } e^{2imt} \langle (a^{\text{out}}_{-k,-s})^\dagger (b^{\text{out}}_{k,s})^\dagger \rangle \\ \text{and } e^{-2imt} \langle b^{\text{out}}_{-k,-s}, a^{\text{out}}_{k,s} \rangle \end{Bmatrix} \tag{7.13}$$

$$\hat{p}_{-1} = \begin{Bmatrix} \text{terms containing } e^{2imt} \langle (a^{\text{out}}_{-k,-s})^\dagger (b^{\text{out}}_{k,s})^\dagger \rangle \\ \text{and } e^{-2imt} \langle b^{\text{out}}_{-k,-s}, a^{\text{out}}_{k,s} \rangle \end{Bmatrix}. \tag{7.14}$$

Both terms oscillate with frequencies $\pm 2m$ in the t time.

The equations above reflect the result of § 5. For $R \to \infty$ corresponding to $\eta \to \infty$ the only remaining term is $\hat{\rho}_0$. It represents according to (5.18) the total rest mass per coordinate volume. The motion of the particles has asymptotically slowed down completely. The terms proportional to R^{-1} can be interpreted as a result of the *zitterbewegung*. They vanish if averaged over times greater than some t_C. In the even more preasymptotic phase, when the R^{-2} terms are to be taken into account, pressure $\hat{p}_{-2}$ and density $\hat{\rho}_{-2}$ of the kinetic energy per coordinate volume are related according to (7.12) by the thermal equation of state of a non-relativistic ideal gas.

Appendix: WKB theory for Dirac particles

A.1. Basic equations

The generally covariant Dirac equation (2.2)

$$i\gamma^{\alpha}\Psi_{\|\alpha} - m\Psi/\hbar = 0, \qquad c = 1 \tag{A.1}$$

may be solved using the expansion in powers of $\hbar$:

$$\Psi_{\delta} = \exp(i\delta S/\hbar) \sum_{n=0}^{\infty} a_n(-i\delta\hbar)^n, \qquad \delta = \pm 1. \tag{A.2}$$

We decompose the spinor a_0 according to

$$a_0 = fb_{\delta} \tag{A.3}$$

with

$$\bar{b}_{\delta} b_{\delta} = \delta \tag{A.4}$$

into a 'normalised' spinor b_{δ} and a real scalar f. The WKB approximation of the Dirac equation is formally obtained by inserting (A.2) into (A.1) and restricting to the term of lowest order in $\hbar$. This implies a Ψ of the form

$$\Psi_{\delta}^{\mathrm{WKB}} = a_0 \exp(i\delta S/\hbar) \tag{A.5}$$

and a differential equation for S and b_{δ}:

$$(\delta\gamma^{\alpha} S_{|\alpha} + m) b_{\delta} = 0. \tag{A.6}$$

Iteration of (A.6) leads with (2.3) to

$$S^{|\alpha} S_{|\alpha} = m^2 \tag{A.7}$$

whence we may infer that

$$S_{|\alpha} = -p_{\alpha} \tag{A.8}$$

with future-poining 4-momentum p_{α}. (A.8) and (A.7) imply

$$p_{\alpha\|\epsilon} p^{3} = 0. \tag{A.9}$$

We insert the WKB approximation (A.5) into the Dirac current j^{α} of (2.4)

$$j^{\alpha} = f^2 \bar{b}_{\delta} \gamma^{\alpha} b_{\delta}. \tag{A.10}$$

This current is still divergence-free in the framework of our approximation. On the other hand, using Gordon's decomposition of the current and again applying the WKB approximation, we are led to

$$j_{\alpha} = -\frac{1}{m} f^2 S_{|\alpha} = \frac{f^2}{m} p_{\alpha} \tag{A.11}$$

which on the level of approximation must be divergence-free as well

$$(f^2 S^{|\alpha})_{\|\alpha} = 0. \tag{A.12}$$

Comparison of (A.10) and (A.11) yields

$$p_{\alpha} = m\bar{b}_{\delta} \gamma_{\alpha} b_{\delta}. \tag{A.13}$$

The WKB norm is given by

$$-\int_{\sigma} \frac{1}{m} f^2 S_{|\alpha} u^{\alpha} \, \mathrm{d}^3 V = 1 \tag{A.14}$$

and agrees because of (A.10) with the Dirac norm (2.5). For the WKB approximation of the energy–momentum tensor (2.6) follows the form of a perfect fluid without pressure

$$T_{\alpha\beta} = \delta f^2 p_\alpha \bar{b}_\delta \gamma_\beta b_\delta = \delta \frac{f^2}{m} p_\alpha p_\beta. \tag{A.15}$$

Hence δ denotes the sign of the energy of the respective solution.

The equations (A.6), (A.7), (A.12) and (A.13) are the basic (differential) equations of the WKB approximation. They allow the determination of S, b and f. We mention that the corresponding WKB approximation of the Klein–Gordon equation leads to (A.7) and (A.12) as well. Of course one may look at the equations (A.4), (A.6), (A.7), (A.12) and (A.13) together with (A.5) as basic rigorous dynamical equations defining a new theory for spin-$\frac{1}{2}$ particles which could be called WKB theory. Note that it does not have a superposition principle. The condition a WKB solution has to fulfill in order also to be an exact Dirac solution is

$$(\gamma^\alpha a_0)_{\|\alpha} = 0. \tag{A.16}$$

A.2. WKB solution

For the line-element (1.1*b*) we find as solution of (A.7)

$$S_{|\alpha} = \begin{cases} -k_{\hat{\alpha}} \\ -\sqrt{(R^2 m^2 + k^2)}, \qquad k^2 = -k_{\hat{\alpha}} k^{\hat{\alpha}}. \end{cases} \tag{A.17}$$

The corresponding solution of (A.12) is

$$f_k = \frac{1}{R} m^{1/2} (2\pi)^{-3/2} (R^2 m^2 + k^2)^{-1/4} \tag{A.18}$$

and the solution of (A.6) is

$$b_{\delta;k,s} = \left(\frac{\hat{E} + mR}{2mR}\right)^{1/2} \begin{bmatrix} 1 & 0 & \dfrac{k_z}{\hat{E}+mR} & \dfrac{k_x - \mathrm{i}k_y}{\hat{E}+mR} \\ 0 & 1 & \dfrac{k_x + \mathrm{i}k_y}{\hat{E}+mR} & \dfrac{-k_z}{\hat{E}+mR} \\ \dfrac{k_z}{\hat{E}+mR} & \dfrac{k_x - \mathrm{i}k_y}{\hat{E}+mR} & 1 & 0 \\ \dfrac{k_x + \mathrm{i}k_y}{\hat{E}+mR} & \dfrac{-k_z}{\hat{E}+mR} & 0 & 1 \end{bmatrix} \tag{A.19}$$

where $\delta, s = \quad 1, 1 \quad 1, -1 \quad -1, 1 \quad -1, -1$

$$\hat{E} = \sqrt{(m^2 R^2 + k^2)}. \tag{A.20}$$

This solution is normalised to unity. We mention that for the domain of negative values of η the arguments of § 2 apply according to (A.10) as well to b:

$$\overline{bR^{3/2}} = \bar{b}R^{3/2}. \tag{A.21}$$

The system of WKB solutions

$$\Psi^{\mathrm{WKB}}_{\delta;\boldsymbol{k},s} = f_k b_{\delta;\boldsymbol{k},s} \exp\left(\mathrm{i}\delta\frac{S_k}{\hbar}\right) \tag{A.22}$$

is complete and allows a decomposition of any normalised Dirac solution according to

$$\Psi^{\mathrm{Dirac}}(x) = \sum_s \int_{\boldsymbol{k}} \mathrm{d}^3k\,(c_{\boldsymbol{k},s}(\eta)\Psi^{\mathrm{WKB}}_{+1;\boldsymbol{k},s}(x) + \mathrm{d}^*_{\boldsymbol{k},s}(\eta)\Psi^{\mathrm{WKB}}_{-1,-\boldsymbol{k},-s}(x)) \tag{A.23}$$

with

$$\sum_s \int_{\boldsymbol{k}} \mathrm{d}^3k\,(|c_{\boldsymbol{k},s}|^2 + |d_{\boldsymbol{k},s}|^2) = 1. \tag{A.24}$$

A.3. WKB *solutions as energy eigenfunctions*

It can be shown that the $\Psi^{\mathrm{WKB}}_{\delta;\boldsymbol{k},s}$ of (A.22) are energy eigenfunctions of the Dirac energy operator of (2.19) corresponding to the energy eigenvalues

$$\tilde{E}_\delta = \delta\hat{E}, \qquad E_\delta = \tilde{E}_\delta/|R|. \tag{A.25}$$

By this the energy eigenfunctions obtain a dynamical interpretation. Moreover using the decomposition (A.23) it follows with (2.18) that

$$\int_\sigma T_{\alpha\beta}u^\alpha u^\beta\,\mathrm{d}^3V = \sum_s \int_{\boldsymbol{k}} \mathrm{d}^3k E_{1;\boldsymbol{k},s}(c^*_{\boldsymbol{k},s}c_{\boldsymbol{k},s} - d_{\boldsymbol{k},s}d^*_{\boldsymbol{k},s}) \tag{A.26}$$

which means that the WKB solutions diagonalise the field-theoretical energy expression. We therefore have that for the space–time in question the particle concept which is based on the diagonalisation of the energy–momentum tensor completely agrees with that of § 3 based on WKB solutions.

A.4. WKB *solutions as exact solutions of the Dirac equation*

We refer again to the space–time (1.1*b*) and the WKB solution with (A.17), (A.18) and (A.19). For $\eta \to \pm\infty$ it can easily be seen that Ψ^{WKB} fulfills the condition (A.16) and is therefore also an exact solution of the Dirac equation. For finite η the validity of condition (A.16) would imply that

$$R^{-5/2}\gamma^{(4)}(R^{3/2}fb)_{|4} = 0 \tag{A.27}$$

with f and b of (A.18) and (A.19). One component of (A.27), because of (A.18) and (A.19), is of the form

$$R^{-5/2}\left[\left(1 + \frac{mR}{(m^2R^2+k^2)^{1/2}}\right)^{1/2}\right]_{|4} = 0. \tag{A.28}$$

For an expansion law

$$R \sim \eta^q, \qquad 0 < q < \infty \tag{A.29}$$

and taking $k \neq 0$ equation (A.28) is fulfilled only for $\eta \to \pm\infty$. Accordingly for all finite values of R and especially for $\eta = 0$ the WKB solutions do not obey the Dirac equation rigorously.

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